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Inviscid quasi-geostrophic flow over topography: testing statistical mechanical theory

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Numerical simulations are employed in a detailed test of the statistical mechanical description of topographic turbulence. Predictions of steady flows correlated with topography are given particular attention. Agreement between numerical and statistical mechanical results is demonstrated for a large range of parameter values, and over an ensemble of random choices of topography and initial conditions.

1. Introduction

Equilibrium statistical mechanics enables the analytical description of statistical properties of inviscid quasi-geostrophic flow over topography (Salmon, Holloway & Hendershott 1976; Frederiksen & Sawford 1981). An important prediction is that eddy–topographic interactions give rise to topographically correlated mean flows. This result has not heretofore been tested in quantitative detail by comparison to numerical computations.

Fresh motivation for examining this problem has arisen from recent suggestions that large-scale geophysical flows may exhibit a tendency to relax toward inviscid absolute equilibria. For example, Holloway (1992) has proposed that ocean models can be modified by replacing the traditional eddy viscosity representation of subgrid-scale effects by a term describing relaxation toward an equilibrium mean flow rather than toward rest. Frederiksen, Dix & Kepert (1995) suggest that the tendency for atmospheric circulation models to exhibit too much energy in their zonal modes stems from a resemblance of the simulated flows to equilibria of the inviscid barotropic vorticity equation. They note that the misrepresentation of the atmospheric energy spectrum actually worsens with increasing resolution, a tendency that is mirrored in the equilibrium spectra. This renewed interest in the connection between geophysical and absolute equilibrium flows has led us to test quantitatively the statistical mechanical results.

2. Results from statistical mechanics

The dimensionless equation of motion for barotropic quasi-geostrophic flow on an f-plane is

$$\partial_t \zeta + \boldsymbol{z} \cdot \nabla \psi \times \nabla(\zeta + h) = 0, \tag{1}$$

which represents conservation of potential vorticity $q = \zeta + h$ by a horizontal velocity field $\boldsymbol{u} = \boldsymbol{z} \times \nabla \psi$, where ψ is the velocity stream function, $\boldsymbol{z}$ is the vertical unit vector, and ∇ is the gradient operator with respect to the horizontal coordinates (x, y). Here $\zeta = \nabla^2 \psi$ is the relative vorticity, and $h(x, y) = f(H_0 - H)/H_0$ is the variation of depth H relative to the mean depth H_0, scaled by the Coriolis parameter f.

In treating equation (1) by the methods of statistical mechanics, the assumption is made that the system is ergodic, i.e. that it visits all locations in phase space accessible under conserved integrals of the motion. We will consider two (second-moment) invariants: energy

$$E = \frac{1}{2}\iint (\nabla\psi)^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

and potential enstrophy

$$Q = \frac{1}{2}\iint q^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{3}$$

Beyond (2) and (3) there is an infinity of invariants given by integrals over the domain (subject to appropriate boundary conditions) of arbitrary functions of potential vorticity. Miller (1990) and Robert & Sommeria (1991) have considered extending statistical mechanical theory for two-dimensional Euler equation flow to include additional invariants. See also Pasmanter (1994) and references therein. In the present context, including topography, we will choose maximally random initial conditions from a Gaussian distribution subject only to second-moment information. This property is preserved as well in the final equilibrium.

Subject to (2) and (3), Salmon *et al.* (1976) construct a microcanonical ensemble for the probability distribution at equilibrium, having expressed the fields in terms of finite eigenfunction expansions. This ensemble yields expressions for the ensemble-mean properties of the system, which correspond to a state of maximum entropy (Carnevale, Frisch & Salmon 1981; Salmon 1982; Holloway 1986). The expressions involve two Lagrange multipliers, α and β, defined implicitly through the relations

$$E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{1}{\alpha+\beta k^2} + \frac{k^2|h_{\boldsymbol{k}}|^2}{((\alpha/\beta)+k^2)^2}, \tag{4}$$

$$Q = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{k^2}{\alpha+\beta k^2} + \frac{(\alpha/\beta)^2|h_{\boldsymbol{k}}|^2}{((\alpha/\beta)+k^2)^2}, \tag{5}$$

where $h_{\boldsymbol{k}}$ is the Fourier coefficient for $h(x,y)$ at wavenumber $\boldsymbol{k}$, and $k = |\boldsymbol{k}|$. Here and in the following a doubly periodic domain is assumed. Note that the individual terms in the summations represent the ensemble-mean spectra of energy and potential enstrophy.

In each of equations (4) and (5), the first term is associated with fluctuations having vanishing ensemble means, and the second term with a temporally steady ensemble-mean flow. The rather surprising implication is that mean flows emerge from random initial conditions when topography is present. For the ensemble-mean stream function, the probability distribution yields

$$\langle\psi_{\boldsymbol{k}}\rangle = \frac{h_{\boldsymbol{k}}}{\mu+k^2}, \tag{6}$$

where $\mu \equiv \alpha/\beta$, and the angled brackets denote an ensemble mean. The steady component of vorticity is thus correlated with topography according to

$$\langle\zeta h^*\rangle_{\boldsymbol{k}} = -\frac{k^2|h_{\boldsymbol{k}}|^2}{\mu+k^2}, \tag{7}$$

where the asterisk denotes complex conjugation. The behaviour of these equilibria in the limit of infinite resolution is discussed by Carnevale & Frederiksen (1987).

3. Comparison with numerical simulations

In the special case $h = 0$, equation (1) describes inviscid two-dimensional turbulence without topography. The corresponding absolute equilibria, first derived by Kraichnan (1967), can be obtained by setting $h_k = 0$ in equations (4) and (5). Note that such equilibria entirely lack a steady component. Agreement with energy spectra deduced from numerical simulations was demonstrated by Fox & Orszag (1973) and Basdevant & Sadourny (1975). Absolute equilibria for two-dimensional flows on a rotating sphere have analogous properties, and have also been shown to agree with numerical solutions (Frederiksen & Sawford 1980).

Including topography enriches the absolute equilibrium solutions by introducing a steady component. New properties are available to test, including the correlation of topography with stream function and vorticity, and the apportionment of E and Q between mean and fluctuating components. However, only very limited tests of statistical mechanical theory have previously been undertaken (Cummins & Holloway 1994; Wang & Vallis 1994). In these studies, scatter plots of $\langle q\rangle$ *vs.* $\langle\psi\rangle$ were constructed in order to illustrate the linear relationship

$$\langle q(x,y)\rangle = \mu\langle\psi(x,y)\rangle, \tag{8}$$

which can be obtained from the inverse Fourier transform of equation (6). In each case, temporal averaging was employed to represent ensemble means. In no instance was the inferred value for μ compared with the theoretical value determined by equations (4) and (5). Thus, only the appearance of linearity of the form (8) was previously examined.

The present work attempts a more comprehensive verification by comparing equations (4)–(7) with spectra obtained from numerical solutions of equation (1) for a doubly periodic domain. The computations employ a Fourier spectral collocation scheme, with 64 collocation points in each of the x- and y-directions. Truncation with respect to $\boldsymbol{k}$ is isotropic, and dealiasing is performed according to the 2/3 rule (Canuto *et al.* 1988). The truncation wavenumber is thus $k_1 = 21.33$. Temporal integration is performed via a leapfrog scheme, with an occasional trapezoidal predictor–corrector step inserted to maintain stability. In all cases, E and Q are conserved to better than one part in 10^3.

In each simulation, random topography is synthesized by selecting normally distributed mode amplitudes according to the one-dimensional power spectrum $(k+k_h)^{-2.5}$, with $k_h = 4$ in units of the fundamental wavenumber. The initial relative vorticity is also specified randomly according to the one-dimensional power spectrum $k^4(k+k_\zeta)^{-6}$, where k_ζ is assigned a range of values. The variance of topography is in each instance set to unity.

Two sets of tests are considered. In the first, 12 separate realizations of random topography and initial conditions are generated, each having $k_\zeta = 5$ and an initial vorticity variance of unity. For each realization, equation (1) is integrated from $t = 0$ to $t = 1200$ or so, with units of time defined by the inverse square root of the initial relative enstrophy. The initial equilibration has largely concluded by $t = 10$. Ensemble-mean spectra for individual realizations are found by temporally averaging the data at 1000 times evenly spaced over the latter three-quarters of the computations. Finally, an ensemble mean is performed by averaging over the various realizations.

This procedure is selected in order that the results depend only upon the statistics of the topography, and not upon the details of any particular topographic realization. Because only the statistics of the topography and initial conditions are specified, the

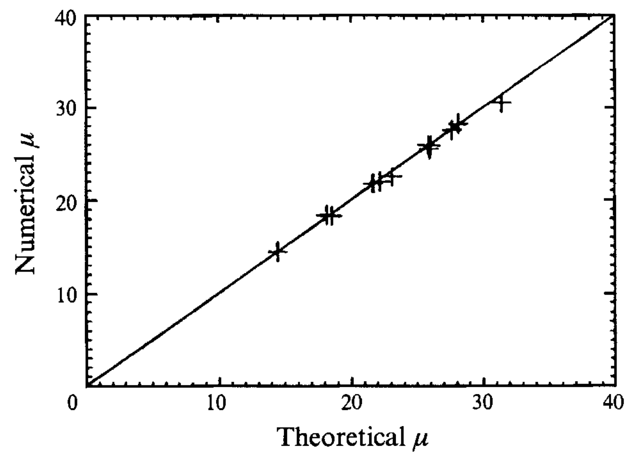

FIGURE 1. Comparison of numerical and theoretical values of Lagrange multiplier ratio μ for each realization in the ensemble.

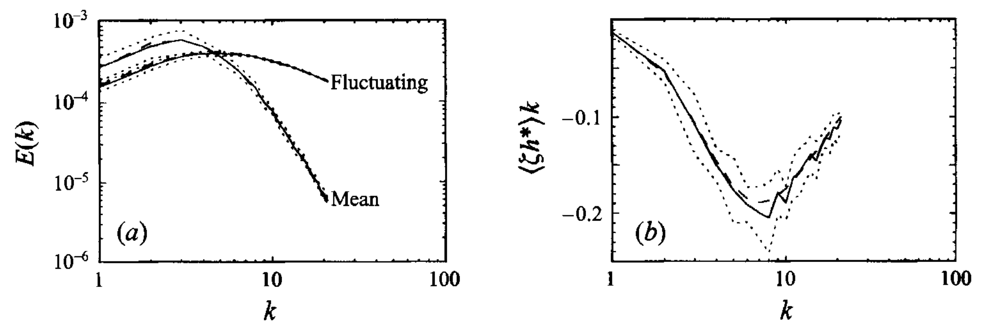

FIGURE 2. Comparison of theoretical and numerical spectra. (*a*) Energy of mean and fluctuating flow components. (*b*) Cross-spectrum of ζ and h, multiplied by k so that the area above the curve corresponds to integrated correlation. Solid lines: numerical ensemble mean spectra for 12 realizations of random topography and initial conditions. Dashed lines: theoretical ensemble mean spectra. Dotted lines: 95 % confidence interval for numerical ensemble mean spectra.

invariants E and Q differ slightly between the individual realizations, with means and standard deviations given by $E = (9.2 \pm 1.2) \times 10^{-3}$ and $Q = 1.00 \pm 0.02$. There is a corresponding spread in the Lagrange multipliers and their ratio μ. Figure 1 shows numerical determinations of μ for each realization, computed from least-squares fits to scatter plots of $\langle q \rangle$ *vs.* $\langle \psi \rangle$. These are compared with theoretical values for μ obtained from a simple bisection search algorithm (e.g. Press *et al.* 1986) that locates α and β satisfying equations (4) and (5). Agreement between numerical and theoretical values for μ is satisfactory for each realization.

The energy spectra of the mean and fluctuating flow components, averaged over the various realizations, are shown in figure 2, along with the averaged vorticity–topography cross-spectrum. The solid lines denote ensemble means and the dotted lines 95 % confidence intervals, based upon variances for the sample of 12 realizations. The dashed lines indicate the predictions of statistical mechanical theory as given by equations (4) and (7), with the h_k described exactly by the power spectrum given above. The theoretical and mean numerical spectra are found to agree to within the numerical confidence intervals.

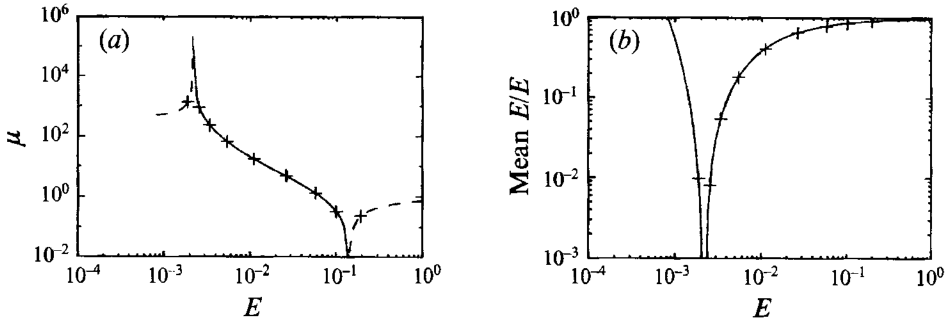

FIGURE 3. Comparison of theoretical and numerical results for fixed Q and topography, with varying E. (*a*) Lagrange multiplier ratio μ, (*b*) fraction of energy in mean flow. The lines represent theoretical results, and the plus sign numerical results. In (*a*) the dashed lines correspond to $\mu < 0$.

In the second set of tests, the topographic variance remains equal to unity, and the initial vorticity variance is tuned so that $Q = 1.00$. E is varied by assigning different values to the scale factor k_ζ, and a single realization of topography is considered. Temporal averaging is performed as above, except that longer averaging periods are chosen in instances where the mean flow is especially weak. Statistical mechanical theory predicts that as E is varied, μ undergoes two changes in sign, corresponding to sign changes of α and β. In figure 3(*a*), positive theoretical values of μ are indicated by the solid lines, and negative values by the dashed lines. The plus signs denote numerical μ determined by the method described above. Note that these values of μ span nearly four orders of magnitude. In figure 3(*b*), theoretical and numerical values for the fraction of energy belonging to the mean flow are similarly compared. In both panels, the theoretical and numerical values are identical to within the plot resolution. Finally, to test that agreement is not restricted to $Q = 1.00$, additional comparisons were made for $Q = 0.75$ and 2.00, with fixed $E = 1.00 \times 10^{-2}$ and the topography as above. Agreement is again obtained.

4. Discussion

How relevant are the inviscid absolute equilibria considered here to large-scale geophysical flows subject to forcing and dissipation, as well as to other effects not represented in equation (1)? Because E and Q are not conserved by such flows, inviscid statistical mechanical theory is not directly applicable to their description. However, numerical experiments indicate that the steady component of ensemble mean flow is relatively unaffected by forcing and dissipation, provided that the corresponding timescales remain longer than the eddy turnover timescale characterizing entropy maximization for all k (Zou & Holloway 1994). (The same is not true for the fluctuating component of flow, because viscosity prevents accumulation of energy near the truncation wavenumber.) That equation (9) approximately holds under such conditions has frequently been demonstrated (e.g. Bretherton & Haidvogel 1976; Treguier 1989; Griffa & Salmon 1989), although this relation apparently can fail for enclosed basins with certain choices of boundary conditions (Cummins 1992; Wang & Vallis 1994). The trademark mean flows of barotropic quasi-geostrophic equilibria also persist when effects of stratification (Salmon *et al.* 1976) and ageostrophy (Griffa *et al.* 1995) are introduced. Based upon these considerations, several recent studies have employed ocean circulation models that include a tendency for relaxation toward absolute equilibrium mean flow (Holloway 1992; Alvarez *et al.* 1994; Eby & Holloway 1994; Holloway, Sou & Eby 1995; Fyfe & Marinone 1995; Sou, Holloway & Eby 1995).

5. Conclusion

Theoretical expressions for the statistical properties of inviscid barotropic quasi-geostrophic flow over topography in a periodic domain have been tested against the results of numerical simulations. The theoretical and numerical results are in excellent agreement for numerous random choices of topography and initial conditions. When topography and potential enstrophy are fixed and energy is varied, close agreement is found over a range of nearly 10^4 in the Lagrange multiplier ratio μ. Statistical mechanical theories that take into account only the quadratic invariants E and Q thus appear to accurately represent equilibrium flow properties when the initial conditions are essentially random.

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